Lecture 14 — Hölder's Inequality

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March 12, 2011

In this lecture, we'll cover several applications of Hölder's Inequality. Before we begin, it's recommended to the reader to be familiar with the following inequalities: Trivial Inequality, Arithmetic Mean - Geometric Mean and the Cauchy-Schwarz Inequality. If the reader is not familiar with these inequalities, it is then advised to read "A Brief Introduction to Inequalites" (Lecture 7) in the OMC archive.

1 The Cauchy-Schwarz Inequality (Generalized)

Let's recall the Cauchy-Schwarz Inequality:

Theorem 1.1 (The Cauchy-Schwarz Inequality)**:** Let $a_1, a_2, \cdots a_n, b_1, b_2, \cdots, b_n$ be real numbers, then,

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

We note that, in the Cauchy-Schwarz Inequality, the left hand side has two products where the terms inside are elevated to the second power. In Hölder's Inequality, we take that two and generalize it. For example, by Hölder's Inequality on positive real numbers $a_1, \cdots, a_n, b_1, \cdots, b_n, c_1, \cdots, c_n$, we have,

$$(a_1^3 + a_2^3 + \cdots + a_n^3)(b_1^3 + b_2^3 + \cdots + b_n^3)(c_1^3 + c_2^3 + \cdots + c_n^3) \geq (a_1b_1c_1 + \cdots + a_nb_nc_n)^3$$

It's important to note that now, instead of there being two products and the terms inside being elevated to the second power, there are three products and the terms inside are cubed. Similarly, if we were to have four products, then the terms inside would be elevated to the fourth power, and so on. Formally, this inequality is equivalent to,

Theorem 1.2 (Hölder's Inequality)**:** For all $a_{i_j} > 0$ where $1 \leq i \leq m$, $1 \leq j \leq n$ we have,

$$\prod_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i_j} \right) \geq \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} a_{i_j}} \right)^m$$

Note that the Cauchy-Schwarz Inequality is Hölder's Inequality for the case $m = 2$.

Example 1.3: Let a, b and c be positive real numbers. Prove that,

$$(a^3 + 2)(b^3 + 2)(c^3 + 2) \geq (a + b + c)^3$$

Proof. By Hölder's Inequality, we have that,

$$(a^3+1+1)(1+b^3+1)(1+1+c^3) \geq \left(\sqrt[3]{a^3\cdot 1\cdot 1}+\sqrt[3]{1\cdot b^3\cdot 1}+\sqrt[3]{1\cdot 1\cdot c^3}\right)^3$$

Or,

$$(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3$$

And we're done! □

Example 1.4: Prove the Arithmetic Mean - Geometric Mean Inequality.

Proof. The Arithmetic Mean - Geometric Mean Inequality states that for positive real numbers $a_1, a_2, \cdots, a_n$ the following inequality holds,

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1a_2a_3\cdots a_n}$$

Hence, it is equivalent to proving that,

$$a_1+a_2+\cdots+a_n \geq n\sqrt[n]{a_1a_2a_3\cdots a_n}$$

Or,

$$(a_1+a_2+\cdots+a_n)^n \geq (na_1a_2\cdots a_n)^n$$

Next we note that,

$$(a_1+a_2+\cdots+a_n)^n = (a_1+a_2+\cdots+a_n)(a_2+a_3+\cdots+a_n+a_1)\cdots(a_n+a_1+a_2+\cdots+a_{n-1})$$

The result then follows directly by applying Hölder's Inequality, and so we are done! □

Example 1.5 (Junior Balkan MO, 2002)**:** Prove that for all positive real numbers a, b, c the following inequality takes place

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}$$

Proof. This problem is probably one of the best examples of Hölder's Inequality. It practically has Hölder's Inequality written all over it. First, we note that $3^3 = 27$, hence we might expect Hölder's Inequality to be used on the product of three terms. Next we note that,

$$2(a+b+c) = (a+b)+(b+c)+(c+a)$$

So, by multiplying both sides of the inequality by $2(a+b+c)^2$, it is equivalent with,

$$((a+b)+(b+c)+(c+a))(b+c+a)\left(\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)}\right) \geq 27$$

Which is true by Hölder's Inequality. Hence the inequality,

$$\frac{1}{b(a+b)}+\frac{1}{c(b+c)}+\frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2}$$

is also true, so we are done! □

Example 1.6: Let a and b be positive real numbers such that $a+b=1$. Prove that,

$$\frac{1}{a^2}+\frac{1}{b^2}\geq 8$$

Proof. First we note that,

$$\frac{1}{a^2}+\frac{1}{b^2}=(a+b)(a+b)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

Then, by Hölder's Inequality, we have,

$$(a+b)(a+b)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\geq\left(\sqrt[3]{\frac{a\cdot a}{a^2}}+\sqrt[3]{\frac{b\cdot b}{b^2}}\right)^3=8$$

□

Example 1.7: Let a,b and c be positive real numbers such that $a+b+c=1$. Prove that,

$$4a^3+9b^3+36c^3\geq 1$$

Proof. Note that,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$$

Then, by applying Hölder's Inequality, we have,

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)(4a^3+9b^3+36c^3)\geq(a+b+c)^3=1$$

And we're done. □

Example 1.8: Let a,b and c be positive real numbers. Prove that,

$$\frac{a+b}{\sqrt{a+2c}}+\frac{b+c}{\sqrt{b+2a}}+\frac{c+a}{\sqrt{c+2b}}\geq 2\sqrt{a+b+c}$$

Proof. A common strategy used when solving problems that include square roots in the denominator is to square the expression on the left hand side then multiply by what's inside the square root times the numerator and apply Hölder's Inequality, like so,

$$\left(\frac{a+b}{\sqrt{a+2c}}+\frac{b+c}{\sqrt{b+2a}}+\frac{c+a}{\sqrt{c+2b}}\right)^2\left(\sum_{cyc}(a+b)(a+2c)\right)\geq 8(a+b+c)^3$$

Next we note that,

$$\sum_{cyc}(a+b)(a+2c)=(a+b+c)^2+3(ab+bc+ac)$$

Hence, it is sufficient to prove that,

$$\frac{8(a+b+c)^3}{(a+b+c)^2+3(ab+bc+ac)}\geq(2\sqrt{a+b+c})^2$$

The rest of the proof is left as an exercise to the reader. □

1.1 Practice Problems

1. Let a, b and c be positive real numbers. Prove that,

   (a)
$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)^3}{3(ab+bc+ac)}$$

   (b)
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sqrt{\frac{27}{ab+bc+ac}}$$

   (c)
$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}$$

   (d)
$$\frac{a^2+b^2+c^2}{a+b+c}\geq\sqrt{\frac{abc(a+b+c)}{ab+bc+ac}}$$

   (e)
$$a^3+b^3+c^3\leq 3 \implies a+b+c\leq 3$$

2. Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that,

   (a)
$$\sqrt[3]{99}\geq\sqrt[3]{1+8a}+\sqrt[3]{1+8b}+\sqrt[3]{1+8c}$$

   (b) For a positive integer n:
$$\sqrt[n]{ab+bc+ac}\geq a\sqrt[n]{\frac{b+c}{2}}+b\sqrt[n]{\frac{a+c}{2}}+c\sqrt[n]{\frac{a+b}{2}}$$

3. Let $a_1, a_2, \cdots, a_n$ be positive real numbers. Prove that,
$$(1+a_1)(1+a_2)\cdots(1+a_n)\geq(1+\sqrt[n]{a_1a_2\cdots a_n})^n$$

4. Let a, b, c, x, y and z be positive real numbers. Prove that,
$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\geq\frac{(a+b+c)^3}{3(x+y+z)}$$

5. Let a, b and c be positive real numbers such that $a+b+c=1$. Prove that,
$$\frac{1}{a(3b+1)}+\frac{1}{b(3c+1)}+\frac{1}{c(3a+1)}\geq\frac{9}{2}$$

6. Let a and b be positive real numbers such that $a^2+b^2=1$. Prove that,
$$\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{b}{a^2+1}+\frac{a}{b^2+1}\right)\geq\frac{8}{3}$$

7. Let a, b and c be positive real numbers. Prove that,

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \leq \sqrt[3]{a \cdot \left(\frac{a+b}{2}\right) \cdot \left(\frac{a+b+c}{3}\right)}$$

8. (Vasile Cirtoaje) Let a, b and c be positive real numbers. Prove that,

$$\frac{a}{\sqrt{a+2b}}+\frac{b}{\sqrt{b+2c}}+\frac{c}{\sqrt{c+2a}} \geq \sqrt{a+b+c}$$

9. (Samin Riasat) Let a, b, c, m, n be positive real numbers. Prove that,

$$\frac{a^2}{b(ma+nb)}+\frac{b^2}{c(mb+nc)}+\frac{c^2}{a(mc+na)} \geq \frac{3}{m+n}$$

10. (USAMO, 2004) For positive real numbers a, b and c. Prove that,

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \geq (a+b+c)^3$$

11. (IMO, 2001) Prove that for all real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1$$

References

1. Pham Kim Hung, *Secrets in Inequalities (Volume 1)*, GIL Publishing House, 2007.

2. Samin Riasat, *Basics of Olympiad Inequalities*, 2008.

3. Radmila Bulajich Manfrino, José Antonio Gómez Ortega, Rogelio Valcez Delgado, *Desigualdades*, Instituto de Matemáticas, Universidad Nacional Autónoma de México, 2005.